A Parallel Finite Element Method for the Pseudostress-Velocity Form Stokes Equations
- Multigrid Solution Built on SAMRAI

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ABSTRACT: We investigate a mixed finite element method and a fast multigrid solution method for the Stokes equation. Our mixed finite element method is based on the pseudostress-velocity formulation. The penalized pseudostress system is solved by a $H(\text{div})$ type multigrid method and the velocity is then calculated explicitly. We also study multigrid convergence rate with variance parallel partitioning and different smoother types.

1 Problem

The pseudostress-velocity formulation for Stokes equation is

$$\begin{cases} -\nabla\cdot\boldsymbol{\sigma} = \mathbf{f} & \text{in } \Omega, \\ \mathcal{A}\boldsymbol{\sigma} - \mu\nabla\mathbf{u} = 0 & \text{in } \Omega \end{cases}$$

the boundary condition is

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \quad \text{in } \Omega.$$

An advantage of this formulation is that the equations for pseudostress and velocity are inherently decoupled, so only a single step without a second solver for projection is needed. By using the penalty method, the pseudostress is computed by solving a symmetric positive definite problem: find $\boldsymbol{\sigma}_h^\epsilon \in RT_0^2$ such that

$$\frac{1}{\mu}(\mathcal{A}\boldsymbol{\sigma}_h^\epsilon, \boldsymbol{\tau}) + \frac{1}{\epsilon}(\nabla\cdot\boldsymbol{\sigma}_h^\epsilon, \nabla\cdot\boldsymbol{\tau}) = -\frac{1}{\epsilon}(\mathbf{f}_h, \nabla\cdot\boldsymbol{\tau}) + \int_{\partial\Omega} \mathbf{g}_h\cdot(\mathbf{n}\cdot\boldsymbol{\tau})dx,$$

for $\forall\boldsymbol{\tau}\in RT_0^2$. The velocity field is then computed explicitly by

$$\mathbf{u}_h^\epsilon = \frac{1}{\epsilon}(\nabla\cdot\boldsymbol{\sigma}_h^\epsilon + \mathbf{f}_h)$$

And we have proved the following error estimate:

$$\|\mathbf{u}-\mathbf{u}_h^\epsilon\| + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h^\epsilon\|_{H(\text{div};\Omega)} \le C\left(h^r(\|\mathbf{u}\|_r + \|\boldsymbol{\sigma}\|_r) + \epsilon(\|\mathbf{f}\| + \|\mathbf{g}\|_{\frac{1}{2},\partial\Omega})\right)$$

2 Multigrid Method

A typical V-cycle multigrid with multiplicative Schwarz smoother is used to solve the pseudostress system. In the following table we present numerical results on accuracy and convergence rate of multigrid solver. The numerical problem we considered is a 2-D problem on the domain $\Omega = (0,1)\times(0,1)$ with exact solution

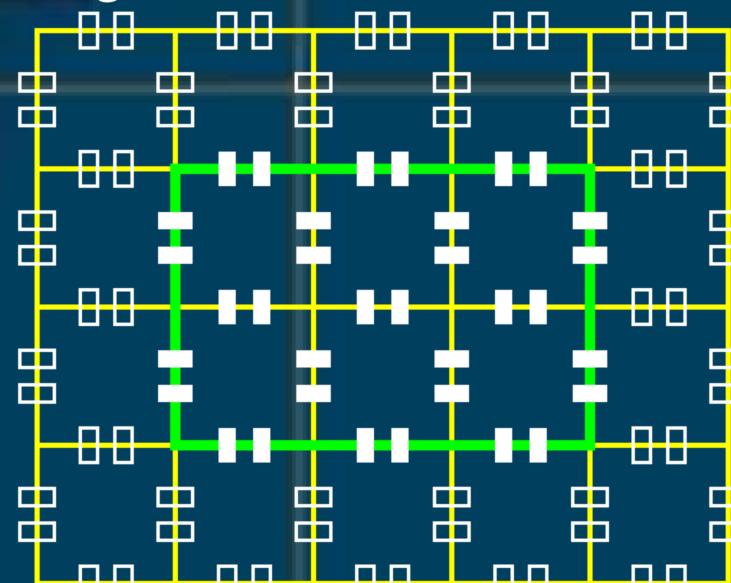
$$\mathbf{u} = \begin{pmatrix} \sin 2\pi x \cos 2\pi y \\ -\cos 2\pi x \sin 2\pi y \end{pmatrix}, \quad \text{and} \quad p = x^2 + y^2.$$

The penalty is taken to be equal to the mesh size.

Size	d.o.f.	Iterations	Conv Rate	$\|\mathbf{u}_h^\epsilon - \mathbf{u}\|$	CPU-time(s)
16*16	1088	10	0.106413	0.00835249	0.37428
32*32	4224	11	0.136397	0.00216851	0.75708
64*64	16640	11	0.145655	0.00057974	2.08337
128*128	66048	11	0.148502	0.00016360	7.18987
256*256	263168	11	0.149497	0.00005028	29.2535
512*512	1050624	11	0.149900	0.00001733	122.322

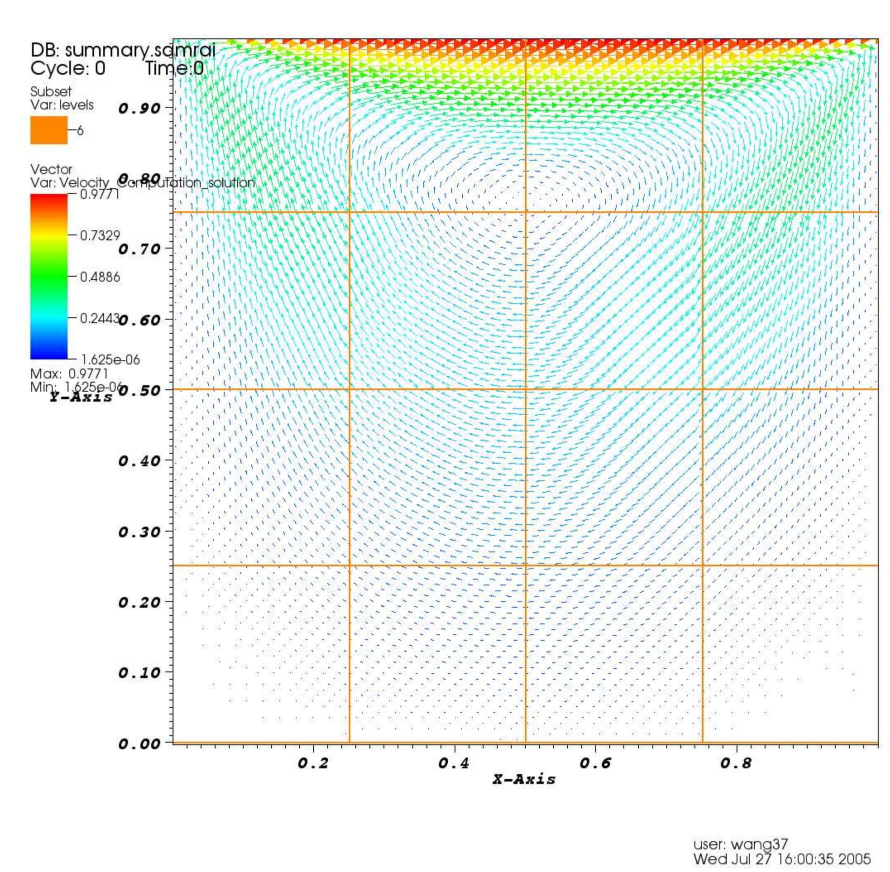
3 Parallel Multigrid Built on SAMRAI

SAMRAI(Structured Adaptive Mesh Refinement Application Infrastructure) employs a dynamic structured PATCH HIERARCHY and individually decomposes each hierarchical level in PARALLEL . Using this framework we developed a parallel multigrid method. (Eventually we want to develop a fast adaptive composite grid method.) To adapt parallel computing, we use red-black type Schwarz Smoother instead of multiplicative Schwarz smoother (Gauss-Seidel type). "Side" data is used and every level is partitioned into several patches with a layer of ghost cells likes the following.



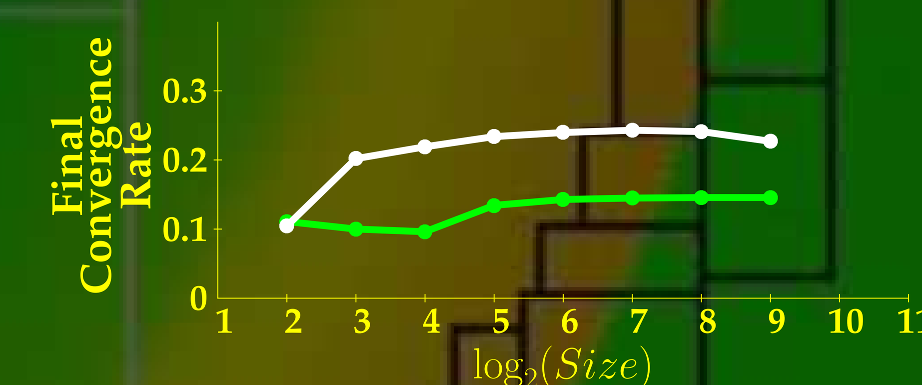
Numerical results of Lid-Driven Cavity Problem is presented in the figure below. The setup of the problem is

- The domain is the square $\Omega = (0,1)\times(0,1)$ uniformly partitioned.
- $\mathbf{f} = 0$, $\mathbf{g}(x,1) \equiv (1,0)$ and $\mathbf{g}(x,y) \equiv 0$ if $y \neq 1$.
- $\mu = 1$.
- The finest level is 128×128 and the biggest patch size is 32×32.



4 Conclusion

- Penalty method decouples pseudostress-velocity formulation such that one can calculate the pseudostress first and then the velocity.
- Physical quantities such as stress, vorticity, and pressure can all be computed algebraically from pseudostress.
- The use of Raviart-Thomas finite elements in discretizing the decoupled pseudostress equation gives a linear system of equations that can be solved efficiently by a parallel multigrid method with red-black type Schwarz smoother. (Final Convergence Rate is around 0.24)
- The convergence rates don't depend on mesh size, and don't depend on parallel partitioning. The following figure shows the detail.



5 Future Work

First, we will focus on FAC method which is a natural extension of conventional multigrid methods applied to problems discretized on adaptive grids.

Second, Navier-Stokes equation for incompressible Newtonian flows will be studied by the same method.

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